Linear Equations and Systems

Linear equations in one variable:

$$ax = b \qquad (a, b\text{: constants})$$

Solution:

$$x = \frac{b}{a} \qquad (a \neq 0)$$

Linear equations in two variables:

$$ax + by = c \qquad (a, b, c\text{: constants})$$

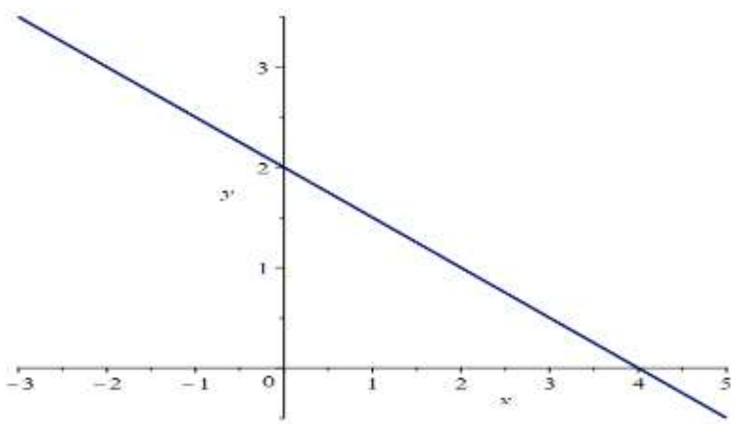
Solutions:

$$\left(x,\ \frac{c - ax}{b} \right) \text{ or } \left(\frac{c - by}{a},\ y \right)$$

Graph:

Line in 2 dimensions

In the 1st solution above x is a free variable, in the 2nd solution y is the free variable.



Linear equations in three variables:

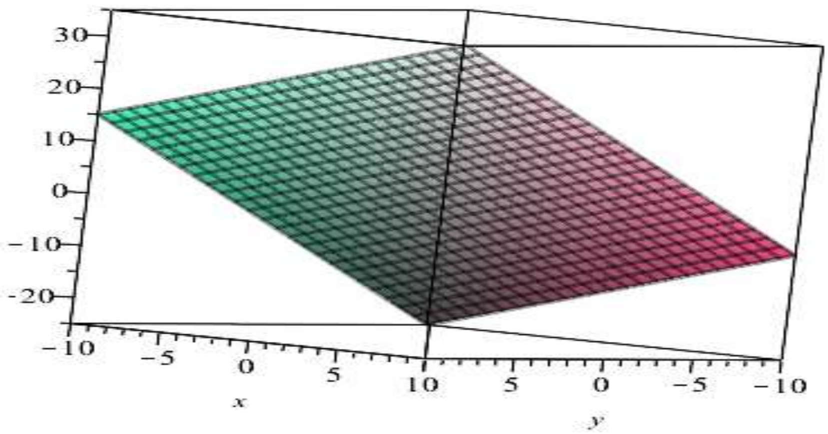
$$ax + by + cz = d \qquad (a, b, c, d\text{: constants})$$

Solutions:

$$\left(x,\ y,\ \frac{d - ax - by}{c} \right) \text{ or } \left(x,\ \frac{d - ax - cz}{b},\ z \right) \text{ or } \ldots$$

Graph:

Plane in 3 dimensions



Linear equations in *n* variables:

$$a_1x_1 + a_2x_2 + a_3x_3 + \cdots + a_{n-1}x_{n-1} + a_nx_n = b \qquad (a_1,\ a_2,\ a_3,\ \ldots\ a_n : \text{constants})$$

Solutions: **Graph:**

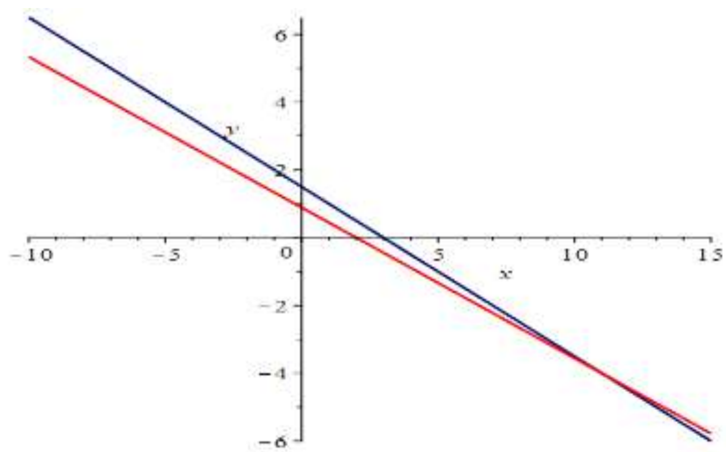
$$\left(x_1,\ x_2,\ x_3,\ \ldots,\ x_{n-1},\ \frac{b - a_1x_1 - a_2x_2 - \cdots - a_{n-1}x_{n-1}}{a_n} \right)$$

Hyperplane in n dimensions

Systems of Linear Equations

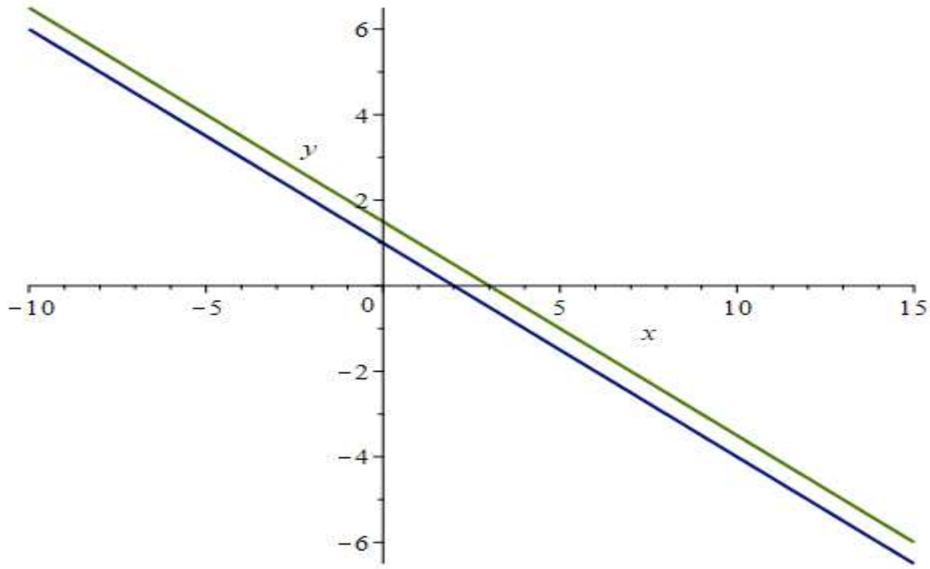
Case I : (**unique solution**)

$$\begin{aligned} x + 2y &= 3 \\ 4x + 9y &= 8 \end{aligned}$$



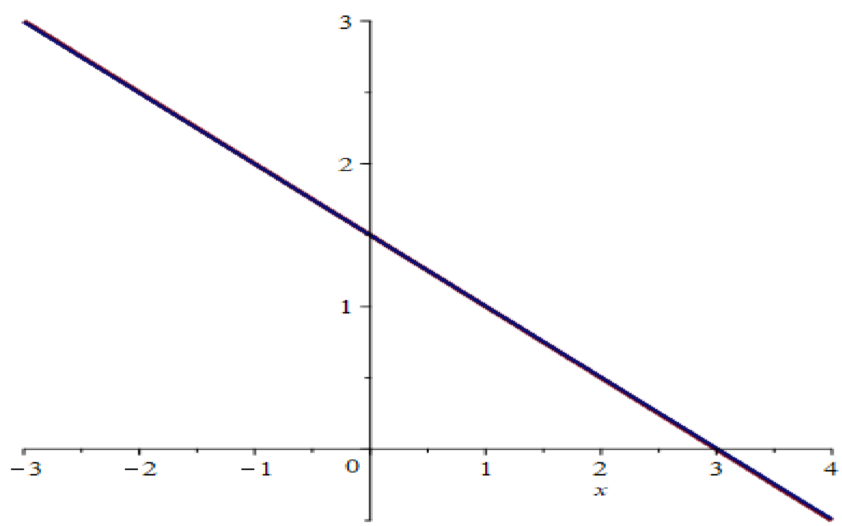
Case II : (**no solution, inconsistent system**)

$$x + 2y = 3$$
$$4x + 8y = 8$$



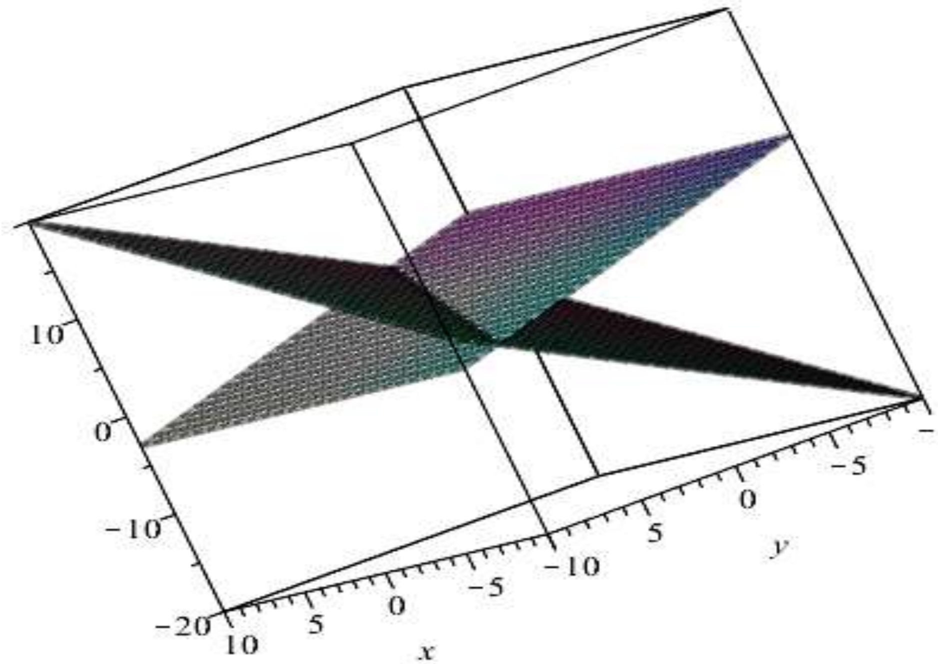
Case III : (**Infinitely Many Solutions**)

$$x + 2y = 3$$
$$4x + 8y = 12$$

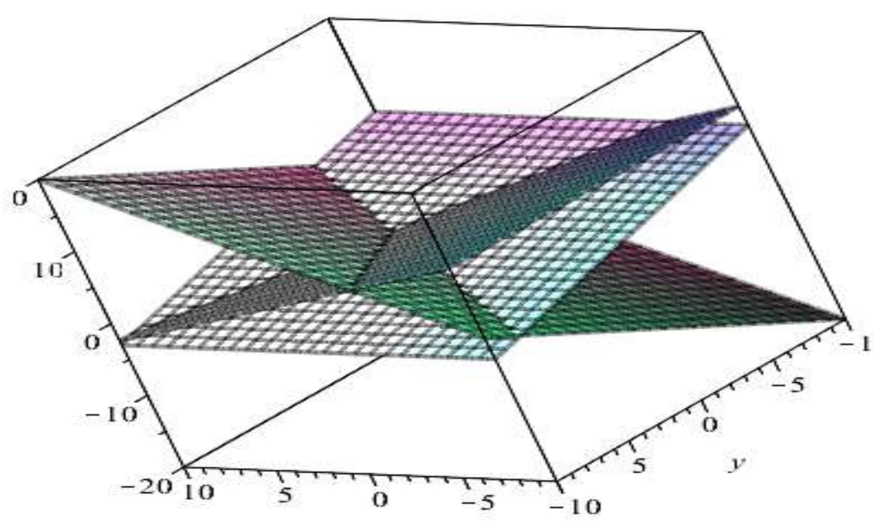


Systems with three variables

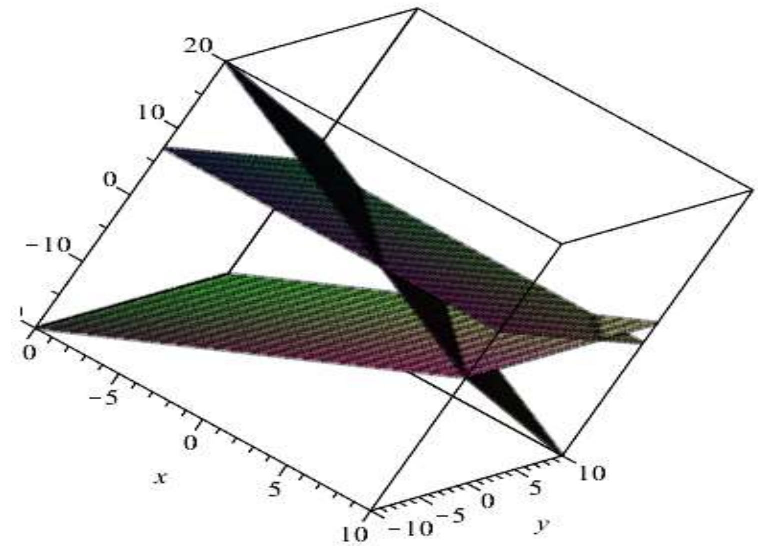
$$\begin{aligned} x + y - z &= 0 \\ -y + 2z &= 4 \end{aligned} \quad \text{(IMS)}$$



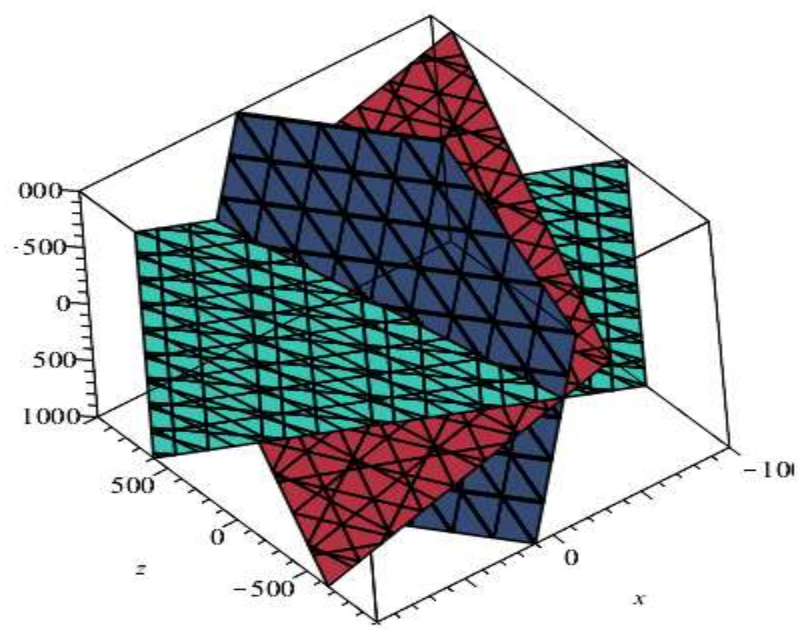
$$\begin{aligned} x + y - z &= 0 \\ -y + 2z &= 4 \\ 3x + 5z &= 18 \end{aligned} \quad \text{(unique solution)}$$



$$\begin{aligned} x+y+z &= 0 \\ y+2z &= 4 \quad \text{(no solution)} \\ x - \quad z &= 10 \end{aligned}$$



$$\begin{aligned} x \;+\; y \;+\; z &= 100 \\ 5x + 10y + 20z &= 2000 \\ -5x \quad + \quad 10z &= 1000 \end{aligned}$$



Homework

In this document there are a total of 7 systems. Below are solutions for 3 of them. Solve the remaining 4 systems and submit your solutions in the beginning of the first lecture.

♦ Solve by substitution.

$$\begin{aligned} x + 2y &= 3 \\ 4x + 9y &= 8 \end{aligned}$$

From the first equation $x = 3 - 2y$. Substituting that into the second equation we get $4(3-2y) + 9y = 8$ or $12 - 8y + 9y = 8$, and therefore $y = -4$ and $x = 11$.

♦ Solve by elimination.

$$\begin{aligned} x + y - z &= 0 \\ -y + 2z &= 4 \\ 3x + 5z &= 18 \end{aligned}$$

If we add the first two equations we eliminate y and we get $x + z = 4$. If we multiply $x + z = 4$ by -3 and add it to equation $3x + 5z = 18$ we get $2z = 6$, therefore $z = 3$, $x = 1$ and $y = 2$.

♦

$$\begin{aligned} x + y - z &= 0 \\ -y + 2z &= 4 \end{aligned}$$

If we add the two equations we get $x + z = 4$ and $x = 4 - z$. From the second equation, $y = 2z - 4$. Therefore, z is a free variable and the solution of the system in parametric form can be written in two different ways,

ordered triple: $(4 - z,\ 2z - 4,\ z)$

vector form: $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 4 - z \\ 2z - 4 \\ z \end{pmatrix}$.

THEOREM 1

The system

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots + a_{1n}x_n &= b_1 \\
a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \cdots + a_{2n}x_n &= b_2 \\
a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + \cdots + a_{3n}x_n &= b_3 \\
\vdots \qquad \vdots \qquad \vdots \qquad\qquad \vdots \quad & \quad \vdots \\
a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 + \cdots + a_{mn}x_n &= b_m
\end{aligned}
\qquad \textbf{(1)}
$$

has either a unique solution, or no solution, or infinitely many solutions.

Proof: (coming soon)